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A new unified, balanced, and conceptual approach to teaching linear algebra

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Abstract

We develop the philosophical and practical background for teaching an elementary Linear Algebra course from the requirements that are particular to the subject. It mixes the inner workings and logic of Linear Algebra and matrices with concepts, hands-on computational schemes, and applications for a satisfying comprehensive teaching and learning experience.
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1. Thoughts

1.1. History

The history of Linear Algebra begins 170 years after that of determinants.

"Matrices" are introduced by Sylvester (1848) and Cayley (1858). They are developed late in the 19th century and come into broad use early in the 20th century. The first textbooks with matrices and Linear Algebra are Perron's [5, vol. 1] and van der Waerden's [8, vol. 2] "Algebra". Our subject occupies short chapters in both of these two-volume sets. Linear Algebra begins to take hold in the undergraduate curriculum after the advent of computers. Halmos' "Finite-Dimensional Vector Spaces" is an early and highly influential graduate level textbook [2]. It is highly abstract and offers neither applications nor concrete algorithms. In the 1960s and 1970s, introductory Linear Algebra courses appeared for undergraduates in the US. An early thorough textbook is by Schneider and Barker [6] that—in retrospect—was ahead of its time in its concrete use of matrices. The typical textbook from the 1970s onwards—there are

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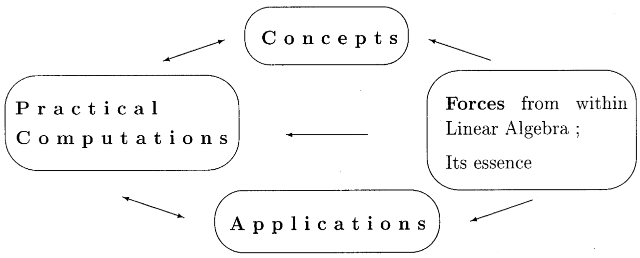
literally hundreds of them—begins with applications and concrete examples and then proceeds to the underlying concepts, for example see Anton's "Elementary Linear Algebra" [1]. This approach has persisted into the present, see Lay's "Linear Algebra and its Applications" [3, 2nd ed., 2000] e.g., The "Linear Algebra Curriculum Study Group Recommendations" [4, p. 42] endorse this approach: "A matrix-oriented linear algebra course should proceed from concrete, and in many cases practical, examples to the development of general concepts, principles, ..."

Well over 100,000 undergraduate students, and possibly as many as 300,000 take an elementary Linear Algebra course in the USA every year. There are at least 3000 college teachers teaching undergraduate Linear Algebra every year in the US.

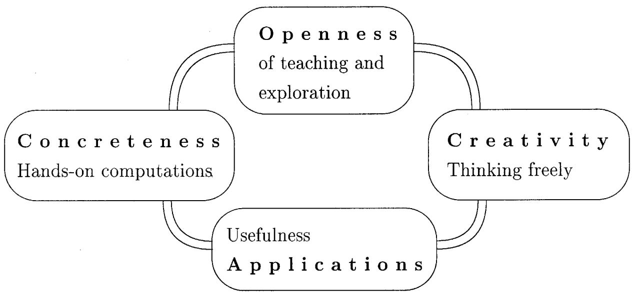
What is Linear Algebra, how does it work, what drives it? How to teach it?

1.2. A comprehensive approach to teaching Linear Algebra

A successful comprehensive approach to teaching Linear Algebra can be developed by finding a balance between the parts of the following diagram:



In our opinion *student learning* and *teacher teaching* in math succeeds best if we rely on the best description, the most complete image and picture of the subject, and adhere to and foster the following interlocking pedagogical principles:



These two sets of four are at the core of learning for most subjects if we adapt the actual boxed notions in both diagrams to reflect the specific field of learning. In particular for the second diagram we note:

- Creativity stimulates thinking and perception.

- Openness appeals to the maturity of the students and gives them the key to the world.
- Concrete computations reinforce concepts and enable comprehension.
  Applications train students to think in frameworks. They show the beauty and the strength of the subject.

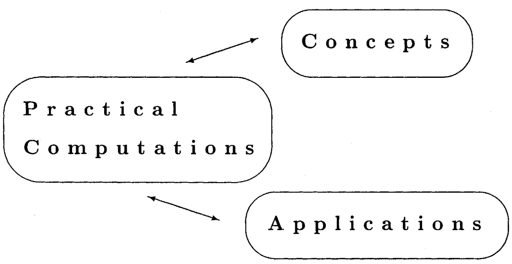
We shall outline the tasks of a Linear Algebra course that honors these two four–fold balances and reinterpret the erstwhile dictum that "a teacher should proceed from the familiar to the unfamiliar, from the concrete to the abstract" (Anton [1, p. vii]).

A course takes place inside a time frame. Time progresses in one direction from the past through the present to the future. We can only teach, practice, and do one thing in real time at any given moment in the real world.

A math course and learning takes place inside a mind, brain, and soul. The human mind has many aspects and layers; it is multi-dimensional and multi-tasking. It processes information and emotions in the background, as well as in the foreground. "Things" become clearer when we actively think about them, and even when we don't think about them, but revisit later. By being conscious of this foreground/background mental process, teachers can outwit the limits of time and its one directional progression through multi-faceted and multi-balanced teaching. An old wives tale says: "don't go to the next chapter unless you fully understand the previous one". This is not correct. One thesis of this paper is that learning math with the goal of comprehension must involve all aspects of a student's mind, the conscious as well as the less conscious; the mundane chore level as well as the free creatively thinking capacity of man. We are capable to fathom both clarity and confusion and we thrive on the creative results coming from the interweaving of both. Learning flourishes when all mental capacities of man are in use. However, this type of learning with its goal of comprehension requires an attempt by the teacher and textbook to convey the fundamental concepts of the subject.

Therefore to teach Linear Algebra (or any other subject) may succeed best if we use a *balanced approach*. This is not a scatterbrained approach. We must explain and rely on the meaningful relationships between the various ingredients from each learning group.

Where are we to start in a twice fourfold balanced presentation of Linear Algebra? We could start with any of the three areas on the left of our initial diagram:



From the 1970s until today, every concrete Linear Algebra textbook based approach starts with practical computations, such as Gaussian row reduction, or with applications such as systems of linear equations and progresses to the underlying concepts. It is driven by concrete forces and attempts to understand abstract concepts from examples.

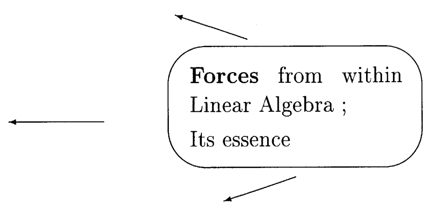
This approach does not take in the more fundamental power and the natural forces that are within Linear Algebra itself. Hence it does not benefit from the essence of its own field of study. To build well with rock, a stonemason must understand the nature of "rock". To train a dog, one must understand the nature and instinct of "dog". As dogs and rocks are different, so is Linear Algebra like no other field or subject of mathematics. The biggest drawback to teaching and learning Linear Algebra with a concrete example driven approach lies in a seeming inability of its students to comprehend the intrinsic power of matrices. Such an approach leads to case thinking, no woods for the trees; lack of student satisfaction, drab facts, no eurekas. It does not lead to a comprehensive understanding and does not give the ability to work independently in and with the subject.

What is missing in this now standard approach?

Conceptual examples and exploratory introductions that enable abstract understanding of our subject. Instead of giving concrete examples, we need to expose the basic structure and the concepts of Linear Algebra from within and build on them.

Why don't we show the secrets of life and of math as we know them to the young?

To communicate the essence of Linear Algebra is quite easy. Linear Algebra can speak strongly and clearly to our students; we just need to let the forces within Linear Algebra guide us. If, instead, we base our teaching on the right most part of the initial diagram



we can develop both "concepts" and "applications" and then bridge the gap, say, between abstract linear transformations and concrete network flow problems by using "computations". Such an approach makes the course vibrant and viable. Some of the concepts and some of the applications will take time to comprehend. But they will be understood.

Most great and successful movies, books, and plays challenge us by frightening and exhilarating us. Why can't we be as successful in math teaching? In every mathematics teaching, thought should be balanced by action, concepts by applications and computations. Not to be afraid of entering the full realm of matrices and Linear Algebra should be our motto.

2. Practice

2.1. Forces from within

Any first course on Linear Algebra deals with many subjects and concepts that are both concrete and abstract and that are mostly new to the students. How can we enter this realm successfully and how can we ensure success in teaching the course?

What we have called "forces from within" may help determine how our course should progress. In this context, the subjects and concepts follow in a certain sequence that is suggested by the framework that lies embedded in Linear Algebra. One could start this course with nearly any of its subjects, such as starting from linear transformations, or row reduction, or linear equations, or bases and basis change, or matrix representations, or determinants for example. In order to convey a coherent understanding of the whole field of Linear Algebra, what comes first dictates to a degree what comes next and how well Linear Algebra is understood in the end. This is due to the natural subject inter-dependencies in Linear Algebra.

For example, if we decide to start with generating vectors, bases and basis change, we would start with the linear dependency implication $\sum_i \alpha_i u_i = 0 \Longrightarrow \alpha_i = 0$. This unconventional, but certainly feasible approach would lead us to study systems of linear equations $\sum_i \alpha_i u_i = 0$, probably followed by row reduction, and then on to the other subjects of Linear Algebra. If, on the other hand, we would rather start with understanding matrix representations of linear transformations, we would possibly start with matrix similarity, then branch into matrix inverses, leading us to Gaussian elimination and row reduction, again followed by linear equations, and so forth. Or we might want to follow history and start with determinants, Laplace expansion and their evaluation via row reductions, linear equations, etc. The inner logic of the field and some of the forces within Linear Algebra drive the sequence of subjects. Loosely speaking, we can start a Linear Algebra course almost anywhere and the inner forces will show and lead us through.

These "inner forces" seem to lead us quickly to linear equations and row reduction from any start. Thus row reduction or Gaussian elimination generally is considered central to elementary Linear Algebra. Seemingly, every recent textbook accounts for this fact and starts with these two subjects. In a sense, the row reduction process is *elemental* to Linear Algebra. It is an essential tool, an algorithm, that lets us compute concrete solutions to elementary linear algebra problems. Thus it is highly commendable that almost all modern textbooks and syllabi introduce this elemental tool of Linear Algebra early. Note that Halmos' 1942/1958 book starts out with Chapter 1 on "Spaces", and that no pre-1965 textbook features row reduction at all.

Starting with row reduction and linear equations, however, seems to limit our students' conceptual comprehension and may ultimately be confusing to students since this approach seemingly precludes a conceptual understanding of the subject. Recall that in the US, typical sophomore students have been exposed to very little that can be construed as "conceptual" in their middle and high school math education, as well as in calculus. Yet Linear Algebra is highly so. To corroborate, there is ample talk in the math ed literature of classes hitting a "brick wall", when linear (in)dependence is studied in the middle of such a course, see several of the contributions in 'The College Mathematics Journal, vol. 24.1, 1993' [4] for examples. In our opinion this 'brick wall' exists—at least in parts—due to an inappropriate elemental start of our syllabi.

We may take a hint from many of Shakespeare's prologues: They name the subject, the fundamental struggle of the play, first lightly; and then the play refers back to its 'theme' and develops it. In this vein, we could do much better when teaching Linear Algebra if we could first touch upon the 'theme', the fundamental concept of Linear Algebra, rather than just the elemental one. We have just said: "the" fundamental concept. Whether such exists, and what it may be if it exists, is not clear. If we could explain most all of the subjects in an elementary Linear Algebra course coherently in a conceptual form, based on this fundament, then our students would benefit tremendously. We would not want to give up the elemental force of 'row reduction' that has entered Linear Algebra teaching so successfully in the last 30+ years. We would, however, have to put it in its proper place in the course and use a two pronged approach instead: rely on row reduction as the elemental algorithm that helps us 'solve' most problems in a concrete numbers sense and rely on its fundamental concept to convey the subject as a whole.

What may this '*fundamental concept*' be? Does such exist for Linear Algebra?

Looking back at our earlier sketches for starting a Linear Algebra course from: (a) basis vectors, (b) representations of linear transformations, (c) linear equations, or (d) determinants, for example, we acknowledge that each of these starting subjects is important in its own right, but only locally so, since none affects all of Linear Algebra.

We do not know all potential subjects and concepts that could be taken as 'fundamental' so that we could build most all of elementary Linear Algebra from each one. We have no such list. At the moment we have only one candidate for this role: "*Linear Transformations*". This notion satisfies our basic requirement of being 'fundamental' to the whole field. Namely, linear transformations can be used to introduce and help explain most every other subject, concept, and area of elementary Linear Algebra. This we shall demonstrate next.[1]

2.2. Details

For a wholistic, unified, and balanced approach to teaching Linear Algebra, we should start with the key notion of Linear Algebra.

In our opinion, Linear Algebra essentially deals with:

vectors, *geometry*, *linear transformations*, and *matrices*.

[1] The book "Linear Functions and Matrix Theory" by B. Jacob, Springer, 1995, is also based on linear transformations and stresses concepts. It nicely covers a small number of standard subjects of a beginning Linear Algebra course.

And in essence, Linear Algebra is governed by an equation, namely the algebraic linearity condition

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y).$$

This equation contains the main ingredients of the subject. Namely

(a) *vectors* x and y,
(b) linear combinations $\alpha x + \beta y$, or *geometry*, for scalars α and β and vectors $x, \ y$,
(c) and a *linear transformation* f between two vector spaces.

Being defined and based on an equation, "Linear Algebra" thus is a natural part of "Algebra". Clearly this fundamental equation should serve well as the conceptual core and the beginning of our studies and teaching.

Therefore one of the first tasks in elementary Linear Algebra consists of describing all linear transformations $f : \mathbb{R}^n \to \mathbb{R}^m$ as matrix $\times$ vector multiplication, where the "standard matrix" $A_{\mathscr{E}}$ for f and the matrix $\times$ vector product are short-hand notations for the action of the linear map f.

This puts matrices firmly at the core of Linear Algebra.

Following our desire for balance, we now balance the conceptual with the concrete in our teaching. For this purpose we introduce row reduction and applications to linear equations. Further balancing is needed, however. Row reduction is mechanically tedious to do by pencil on paper. To learn and understand this algorithm, it suffices to practice it over the integers. Hence each teacher must learn how to construct infinitely many integer test problems. When this relatively complicated algorithm, 'relatively complicated' in relation to the math maturity of sophomores, has been balanced with easy integer arithmetic, students realize quickly that linear transformations (or matrices), row reduction, and linear equations are intimately linked.

The geometry of $\mathbb{R}^n$ extends, however, beyond mere vectors: Every linear transformation generates two intrinsic and complementary subspaces: the image and kernel. Any description of the image involves the solvability of a linear system $Ax = b$, while every solution to $Ax = 0$ belongs to the kernel. That is, both types of subspaces can be well understood from linear equations and linear transformations $x \mapsto Ax$. To continue, we may ask: how large are these two elemental subspaces, how can we describe them. One, the image, is a span, while the kernel obeys a set of defining equations. We can translate between these two generic descriptions of a subspace by using special row reduction schemes, and thus we are back to computations.

Subspaces lead to the "king chapter" of any elementary Linear Algebra course, to

linear (in)dependence, basis, and dimension.

The 'classical' and standard first definition of linear independence of column vectors u_i:

$$\sum_i \alpha_i u_i = 0 \implies \text{all } \alpha_i = 0$$

should, however, come only after a transformation based one. We rather introduce linear (in)dependence based on the vector–matrix identification:

$$k \text{ vectors } u_i \in \mathbb{R}^n \quad \longleftrightarrow \quad U = \begin{pmatrix} | & & | \\ u_1 & \dots & u_k \\ | & & | \end{pmatrix}_{nk},$$

the matrix with columns $u_i \in \mathbb{R}^n$.

Now we study the linear transformation $x \in \mathbb{R}^k \mapsto Ux \in \mathbb{R}^n$: How large is the image of $\mathbb{R}^k$ under the mapping by U? What vectors generate $\operatorname{im}(U) = \operatorname{span}\{u_1, \dots, u_k\}$?

A row reduction R of U shows that any column vector u_i without a corresponding pivot in R is a linear combination, i.e., linearly dependent of the previous columns u_j for $j < i$.

Hence our *preferred* first concrete linear independence definition:

A set of vectors $u_1, \dots, u_k \in \mathbb{R}^n$ is

linearly dependent $\iff$ a row echelon form of U_{nk} has less than k pivots;

It is linearly independent $\iff$ a row echelon form of U_{nk} has k pivots.

By applying the unique solvability criterion of linear systems (no free columns) we obtain the 'classical' linear (in)dependence condition. This gives us a dual insight: computing a row echelon form R of the column vector matrix U decides linear independence among the u_i practically, while the 'classical' definition helps in abstract settings such as in proofs.

In the same vein we develop a dual definition for the concept of a 'basis', namely: a basis of a (sub)space is defined in two equivalent ways as

(a) a maximally linearly independent set of vectors in that (sub)space, or
(b) a minimal spanning set of vectors for the (sub)space.

Knowing 'basis' leads us to study "basis change". Here we again identify a given basis $\mathscr{U} = \{u_1, \dots, u_n\}$ of $\mathbb{R}^n$ with the column vector matrix

$$U = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix}_{nn}.$$

For the standard unit vector basis $\mathscr{E} := \{e_1, \dots, e_n\}$, a point $x = \sum_i x_i e_i \in \mathbb{R}^n$ has the standard $\mathscr{E}$-basis coordinate vector

$$x = x_{\mathscr{E}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

For a basis $\mathscr{U} = \{u_1, \dots, u_n\}$ of $\mathbb{R}^n$, the point $x = \sum_i \beta_i u_i \in \mathbb{R}^n$ has the $\mathscr{U}$-coordinate vector

$$x_{\mathscr{U}} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} \quad \text{with } x = Ux_{\mathscr{U}}.$$

For any third basis $\mathscr{V} = \{v_1, \ldots, v_n\}$ we likewise have $x = x_{\mathscr{E}} = Vx_{\mathscr{V}}$. Thus

$$Vx_{\mathscr{V}} = x = Ux_{\mathscr{U}}, \quad \text{or} \quad x_{\mathscr{V}} = V^{-1}Ux_{\mathscr{U}} \quad \text{and} \quad x_{\mathscr{U}} = U^{-1}Vx_{\mathscr{V}}.$$

This points to a practical row reduction scheme for finding basis change matrices $X_{\mathscr{V}\leftarrow\mathscr{U}} := V^{-1}U$. To be able to compute these easily by hand, teachers must be made familiar with generating unimodular integer matrices.

Everything mentioned and practiced so far ties together when we represent a given linear transformation $f : \mathbb{R}^n \to \mathbb{R}^n$, or its standard matrix representation $A = A_{\mathscr{E}}$, with respect to another basis $\mathscr{U}$. The $\mathscr{U}$ basis representation $A_{\mathscr{U}}$ of f maps $\mathscr{U}$-coordinate vectors to $\mathscr{U}$-coordinate vectors, while $A_{\mathscr{E}}$ maps standard $\mathscr{E}$ vectors to standard $\mathscr{E}$ vectors. As $x_{\mathscr{E}} = Ux_{\mathscr{U}}$ and $y_{\mathscr{U}} = U^{-1}y_{\mathscr{E}}$ for all x and $y \in \mathbb{R}^n$, we note that $A_{\mathscr{E}}x_{\mathscr{E}} = A_{\mathscr{E}}Ux_{\mathscr{U}}$ and thus "$A_{\mathscr{U}}$" $x_{\mathscr{U}} = U^{-1}A_{\mathscr{E}}Ux_{\mathscr{U}}$ describes the linear transformation completely in terms of $\mathscr{U}$-coordinate vectors. This interprets a basis change as matrix similarity. It links matrix simplification, and specifically diagonalizability, to the notion of an eigenvector and eigenvalue: if $A_{\mathscr{E}}u_i = \lambda_i u_i$ for n linearly independent vectors u_i, then for the column vector matrix

$$U = \begin{pmatrix} | & & | \\ u_1 & \ldots & u_n \\ | & & | \end{pmatrix},$$

the matrix $A_{\mathscr{U}} = U^{-1}A_{\mathscr{E}}U = \mathrm{diag}(\lambda_i)$ has a most simple diagonal appearance. How do we best tackle the matrix eigenvalue problem, knowing the 'two component glue' of linear transformations and row reduction that holds Linear Algebra together?

Historical approach

- Use the characteristic polynomial and determinants: $\det(\lambda I - A) = 0$.
- Its roots in $\mathbb{C}$ are the eigenvalues of A.

Via linear transformations

- Use vector iteration:
- Form the sequence of vectors $0 \neq y,\ Ay,\ A^2y = A(Ay), \ldots, A^ny$.
- Find the first linear dependency among these iterates via row reduction and thereby the vanishing polynomial $p_y(A)$ for y.
- Its roots are eigenvalues of A.

All eigenvalues of A can be found from the least common multiple of the set of all vanishing polynomials $\{p_{x_i}(A)\}$ for a basis $\{x_1, \ldots, x_n\}$ of $\mathbb{R}^n$. (Of course, with hand-computations, we generally practise this only for $n \leqslant 4$ or 5.) The vector iteration approach leads immediately to the strong version of the Cayley–Hamilton theorem. Vector iteration prepares students naturally for invariant subspaces and the Jordan Normal Form, all in a first course.

For more specific applications we turn to orthogonality. This can be explained by using the stable modified Gram-Schmidt process in analogy to Gaussian elimination, rather than by using its unstable 'classical' variant. [2] (As teachers we need to be above board and not clutter our student's perception with obsolete algorithms that may take years to correctly dismiss.) Most simply said, the process of orthogonalizing k row vectors $u_1, \ldots, u_k \in \mathbb{R}^n$ in levels via modified Gram–Schmidt is analogous to row reducing the matrix

$$A = \begin{pmatrix} - & u_1 & - \\ & \vdots & \\ & u_k & - \end{pmatrix}_{kn}$$

via Gaussian elimination.

If the complete row reduction of A to row echelon form R can proceed without any swaps, the first level sweep of row reduction creates row equivalent rows $\widetilde{u}_j$ for $j = 2, \ldots, k$ in A's update that lie in the coordinate plane span$\{e_2, \ldots, e_n\} \subset \mathbb{R}^n$. Geometrically, the row reduction of one row u_j via the pivot $u_{11} \neq 0$ of row u_1 projects u_j onto span$\{e_2, \ldots, e_n\}$. On the next level, Gauss uses $\widetilde{u}_2$ to update each of the updated rows $\widetilde{u}_j \in$ span$\{e_2, \ldots, e_n\}$ to lie in span$\{e_3, \ldots, e_n\}$ of $\mathbb{R}^n$ for $j = 3, \ldots, k$, etc.

That is to create a row echelon form R for A via Gauss we first alter $n-1$ row vectors in A via A's first row u_1, then we update the trailing and updated $n-2$ rows of A via its second updated row, etc. (Fig. 1).

The modified Gram–Schmidt algorithm also updates the row vectors $u_1, \ldots, u_k$ of A in levels, but with an eye on orthogonality rather than on zero leading coefficients (Fig. 2).

In modified Gram–Schmidt we first replace each u_j for $j = 2, \ldots, k$ by a vector $v_j \in$ span$\{u_1, u_j\}$ that lies in the $n-1$ dimensional subspace

$$u_1^{\perp} = \{v \in \mathbb{R}^n \mid v \cdot u_1 = 0\}.$$

Then we update the newly computed vectors $v_3, \ldots, v_n$ to lie in span$\{v_2, v_j\}$ and in $v_2^{\perp}$ for $j = 3, \ldots, k$, etc., until we normalize.

Further topics such as symmetric and normal matrices, the singular value decomposition, and the Jordan normal form may lie beyond the reach of a one-semester elementary Linear Algebra course. However, they can now be treated easily by matrix theoretic means such as via the Schur Normal form and vector iteration.

[2] For a well written account of these phenomena, we recommend to look at Lectures 7–9 of "Numerical Linear Algebra" by L. N. Trefethen and D. Bau, SIAM, 1997, where two 8 line MATLAB codes and an enlightening experiment are described.

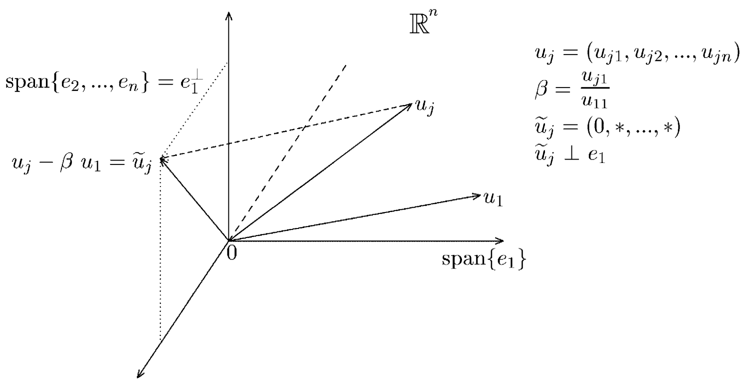


Fig. 1.

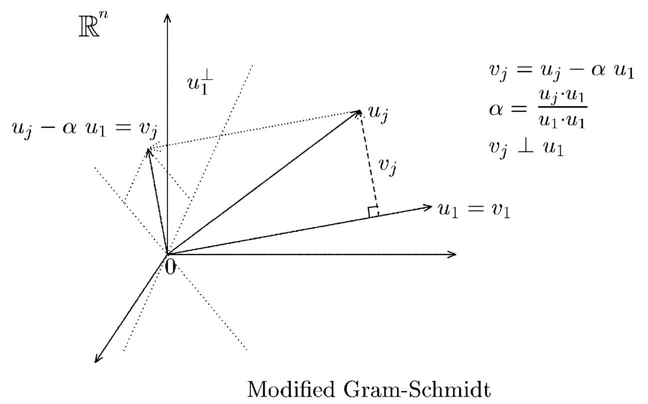


Fig. 2.

3. Results

Rethinking and retooling how to teach elementary Linear Algebra was Emily Haynsworth's desire when I joined her at Auburn. May be I could try to reshape our "miserable 266" introductory course, she suggested in 1982.

The above reformulation of how to teach Linear Algebra from a linear algebraic, unified, balanced, and conceptual viewpoint took 18 years to realize. It has resulted in the textbook "*Transform Linear Algebra*" [7].

Successes. By following a well balanced conceptual approach such as ours, students gain math maturity and teachers find satisfaction in their teaching, i.e., no more 'miserable 266' classes.

The approach emphasizes the value of concepts and first principles, making problem understanding and problem solving easy and possible, perhaps for the first time in a student's math career. The entire course becomes a self-validating experience for students and teachers.

Students taught conceptually from first principles generally become ready tutors of classmates in other parallel sections taught by example driven methods and textbooks based on elemental row reduction.

Complaints from students typically last less than 2 weeks during the initial conceptual 'linear transformation', 'Riesz lemma' phase. Relief sets in when practical row reductions and then linear systems are covered, the latter conceptually at first. Student perception turns to understanding by week 5 or 6 when linear transformations help through 'linear independence' and 'basis'. Most students become downright sophisticated when eigenvalues are introduced and computed by hand via vector iteration. A conceptual approach that is based on linear transformations and row reductions, offers a solid and good learning experience and leads to ever increasing proficiency in math.

Student situation
Pre-1960.

- Relatively few students, Linear Algebra is an upper level or graduate course.
- Relatively high student sophistication and preparation.
- Little need for sophisticated and reflected teaching.

2000s.

- Many students, a sophomore level course.
- A generally low level of preparation (and sometimes the lack of motivation) requires a high level of teacher awareness and reflected teaching.
- *Main task.* To give students confidence in their own thinking power and prowess.
- In our opinion, this mandates a thought–out and conceptual approach to math teaching.

Forces from within Linear Algebra. Linear Algebra has a high level of internal structure. These inner forces drive our chosen sequence of subjects and determine our depth of conceptualization. Linear transformations can be used to act as the fundamental concept and basis for the whole course. When the structure is exposed and real world applications are solved through conceptual understanding, we serve the students well in their intellectual and personal maturation. When we teach from examples, students tend to become disoriented and confused. They often cannot retain concepts long enough to be able to apply them later.

Failures. This approach is difficult for and exposes the 'thinking impaired', 'cannot read', 'no time to study' students. Such learning disorders—around 8–15% of the students now suffer from them in any of my typical undergraduate math courses—become very obvious during a transform based Linear Algebra course.

Acknowledgements

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